Discrete Mathematics: Combinatorics and Graph Theory

Exam 1

Instructions. Solve any 5 questions and state which 5 you would like graded. Write neatly and show your work to receive full credit. You must sign the attendance sheet when returning your booklet. Good luck!

1. Prove or disprove the following:
   (a) For all $m, n \in \mathbb{Z}$, if mn are odd, then m and n are odd.
   (b) There are no integer solutions to the equation $x^2 + 19 = y^2 + 2021$.
   (c) Let $n \in \mathbb{N}$, $n > 1$. If n is not prime then $2^n - 1$ is not prime. Hint: you may use the identity $(a^x - 1) = (a-1)(a^{x-1} + a^{x-2} + \cdots + a^1 + 1)$.

2. Prove the following by induction. State base case, inductive hypothesis and inductive step explicitly.
   (a) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n}(2i-1) = n^2$
   (b) For all $n \in \mathbb{N}$, $n \geq 4$, $3^n \geq n^3$
   (c) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

3. Use the Euclidean Algorithm to find the greatest common divisor of 143 and 33. Use the Extended Euclidean Algorithm to find the Bezout coefficients x, y and all integer solutions to the equation $143x + 33y = gcd(143, 33)$.

4. Prove the following by induction. State base case, inductive hypothesis and inductive step explicitly.
   (a) For all $n \in \mathbb{N}$, $n \geq 2$, $\prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$
   (b) For all $n \in \mathbb{N}$, $7 | (2^{n+2} + 3^{2n+1})$

5. Verify whether the following functions define bijections.
   (a) Let $A = \{x \in \mathbb{R} : x \neq -2\}$ and $B = \{x \in \mathbb{R} : x \neq 1\}$ Is the function $f : A \to B$ defined by $f(x) = \frac{x-2}{x+2}$ (i) injective (ii) surjective and (iii) bijective? Prove or disprove.
   (b) Is the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ (i) injective (ii) surjective and (iii) bijective? What if $f : \mathbb{Q} \to \mathbb{Q}$ or $f : \{-1, 0, 2\} \to \{-1, 0, 8\}$? Prove or disprove.

6. Define C_i by the equations $C_0 = 1$, $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$. Write out the first few terms in the sequence. Prove via strong induction $\forall n > 0$, $C_n \leq n^n$.

7. Construct a bijection $f : [a, b) \to [0, 1)$. Show that f is one-to-one and onto.